Discrete Applied Mathematics 158 (2010) 1205–1211

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Boundary defensive k-alliances in graphs ☆

I.G. Yero, J.A. Rodríguez-Velázquez *

Department of Computer Engineering and Mathematics, Rovira i Virgili University, Av. Països Catalans 26, 43007 Tarragona, Spain

ARTICLE INFO

Article history:
Received 10 May 2008
Received in revised form 22 January 2010
Accepted 12 March 2010
Available online 7 April 2010

Keywords:
Alliances in graphs
Defensive alliance
Domination
Algebraic connectivity
Laplacian spectral radius

ABSTRACT

We define a boundary defensive k-alliance in a graph as a set S of vertices with the property that every vertex in S has exactly k more neighbors in S than it has outside of S. In this paper we study mathematical properties of boundary defensive k-alliances. In particular, we obtain several bounds on the cardinality of every boundary defensive k-alliance. Moreover, we consider the case in which the vertex set of a graph can be partitioned into boundary alliances, showing that if a d-regular graph G of order n can be partitioned into two boundary defensive k-alliances X and Y, then $|X| = |Y| = \frac{n}{2}$ and the algebraic connectivity of G is equal to $d - k$.

1. Introduction

The mathematical properties of alliances in graphs were first studied by Kristiansen, Hedetniemi and Hedetniemi [8]. They proposed alliances of different types: namely, defensive [6–9,12,16], offensive [3,9,10,13,17] and dual or powerful alliances [1,2,18]. For instance, a defensive alliance of a graph G is a set S of vertices of G with the property that every vertex in S has at most one more neighbor outside of S than it has in S. A generalization of defensive alliances was presented by Shafique and Dutton in [14,15] where they define a defensive k-alliance as a set S of vertices of G with the property that every vertex in S has at least k more neighbors in S than it has outside of S. In this paper, we study the mathematical properties of a particular case of k-alliances that we call boundary k-alliances: we define a *boundary defensive k-alliance* in G as a set S of vertices of G with the property that every vertex in S has exactly k more neighbors in S than it has outside of S. We obtain several bounds on the cardinality of every boundary defensive k-alliance. Moreover, we consider the case in which the vertex set of a graph can be partitioned into boundary alliances, showing that if a d-regular graph G of order n can be partitioned into two boundary defensive k-alliances X and Y, then $|X| = |Y| = \frac{n}{2}$ and the algebraic connectivity of G is equal to $d - k$.

We begin by stating the terminology used. Throughout this article, $G = (V, E)$ denotes a simple graph of order $|V| = n$ and size $|E| = m$. We denote two adjacent vertices u and v by $u \sim v$. For a nonempty set $X \subseteq V$, and a vertex $v \in V$, $N_X(v)$ denotes the set of neighbors that v has in X: $N_X(v) := \{u \in X : u \sim v\}$ and the degree of v in X is denoted by $\delta_X(v) = |N_X(v)|$. We denote the degree of a vertex $v_i \in V$ by $d(v_i)$ (or by d_i for short) and the degree sequence of G by $d_1 \geq d_2 \geq \cdots \geq d_n$. The subgraph induced by $S \subset V$ is denoted by $\langle S \rangle$ and the complement of the set S in V is denoted by $\bar{S}$. Moreover, $\partial(S)$ denotes the set of neighbors of S: $\partial(S) = \bigcup_{v \in S} N_{\bar{S}}(v)$. We recall that a set $S \subset V$ is a *dominating set* in G if for every vertex $u \in \bar{S}$, $\delta_S(u) > 0$ (every vertex in $\bar{S}$ is adjacent to at least one vertex in S).

With the above notation we define a defensive k-alliance as follows.

☆ This work was partly supported by the Spanish Ministry of Education through projects TSI2007-65406-C03-01 "E-AEGIS" and CONSOLIDER CSD2007-00004 "ARES" and by the Rovira i Virgili University through project 2006AIRE-09.

* Corresponding author.
E-mail addresses: ismael.gonzalez@urv.cat (I.G. Yero), juanalberto.rodriguez@urv.cat (J.A. Rodríguez-Velázquez).

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doi:10.1016/j.dam.2010.03.005

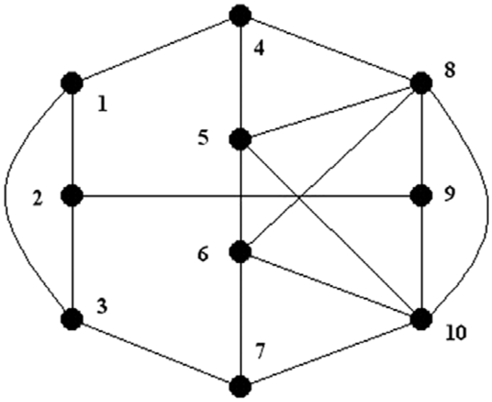

Fig. 1. $S = \{1, 2, 3\}$ is a boundary defensive (1)-alliance.

Definition 1. A set $S \subseteq V$ is a *defensive k-alliance* in $G = (V, E)$, $k \in \{-d_1, \dots, d_1\}$, if

$$\delta_S(v) \geq \delta_{\bar{S}}(v) + k, \quad \forall v \in S. \tag{1}$$

A defensive k-alliance in G is global if it forms a dominating set in G.

In this article we study the limit case of condition (1).

2. Boundary defensive k-alliances

Definition 2. A set $S \subset V$ is a boundary defensive k-alliance in G, $k \in \{-d_1, \dots, d_1\}$, if

$$\delta_S(v) = \delta_{\bar{S}}(v) + k, \quad \forall v \in S. \tag{2}$$

A boundary defensive k-alliance in G is global if it forms a dominating set in G.

In Fig. 1 we show a graph G in which the set $S = \{1, 2, 3\}$ is a boundary defensive (1)-alliance. Notice that this graph does not contain any boundary defensive (0)-alliance.

Remark 1. Let G be a simple graph and let $k \in \{-d_1, \dots, d_1\}$. If for every $v \in V$, $d(v) - k$ is an odd number, then G does not contain any boundary defensive k-alliance.

Corollary 2. *Let G be a d-regular graph and let $k \in \{-d, \dots, d\}$. If $d - k$ is odd, then G does not contain any boundary defensive k-alliance.*

Definition 3. A set $S \subseteq V$ is an *offensive k-alliance* in $G = (V, E)$, $k \in \{2 - d_1, \dots, d_1\}$, if

$$\delta_S(v) \geq \delta_{\bar{S}}(v) + k, \quad \forall v \in \partial(S). \tag{3}$$

An offensive k-alliance in G is global if $\partial(S) = \bar{S}$.

The limit case of condition (3) leads to the concept of *boundary offensive k*-alliances. The case of sets which are both boundary offensive k-alliances and boundary defensive k-alliances was studied in [18].

Remark 3. If S is a defensive k-alliance in G and $\bar{S}$ is a global offensive $(-k)$-alliance in G, then S is a boundary defensive k-alliance in G.

Theorem 4. *Let $G = (V, E)$ be a graph and let $S \subset V$. Let $m\langle S\rangle$ be the size of $\langle S\rangle$ and let c be the number of edges of G with one endpoint in S and the other endpoint outside of S. If S is a boundary defensive k-alliance in G, then*

(i) $m\langle S\rangle = \frac{c+|S|k}{2}$.
(ii) *If G is a d-regular graph, then* $m\langle S\rangle = \frac{|S|(d+k)}{4}$ *and* $c = \frac{|S|(d-k)}{2}$.

Proof. If S is a boundary defensive k-alliance in G, then

$$2m\langle S\rangle = \sum_{v\in S} \delta_S(v) = \sum_{v\in S} \delta_{\bar{S}}(v) + |S|k = c + |S|k.$$

Thus, (i) follows. Moreover,

$$d(v) = 2\delta_{\bar{S}}(v) + k, \quad \forall v \in S.$$

Hence,

$$\sum_{v\in S} d(v) = 2\sum_{v\in S}\delta_{\bar{S}}(v) + |S|k = 2c + |S|k.$$

Therefore, if G is d-regular, $d|S| = 2c + |S|k$. Thus, (ii) follows. $\square$

The bounds shown in Theorem 5 have been obtained in [11], on the minimum cardinality of a standard defensive k-alliance.

Theorem 5. *If S is a boundary defensive k-alliance in a graph G, then*

$$\left\lceil \frac{d_n + k + 2}{2} \right\rceil \le |S| \le \left\lfloor \frac{2n - d_n + k}{2} \right\rfloor.$$

Proof. As $S \subseteq V$ is a boundary defensive k-alliance in G,

$$\frac{d(v) + k}{2} = \delta_S(v) \le |S| - 1, \quad \forall v \in S.$$
$$\frac{d_n + k + 2}{2} \le |S|.$$

Hence, the lower bound follows. On the other hand, if S is a boundary defensive k-alliance in G, then

$$\frac{d_n - k}{2} \le \frac{d(v) - k}{2} = \delta_{\bar{S}}(v) \le n - |S|, \quad \forall v \in S.$$

Thus, the upper bound follows. $\square$

As the following corollary shows, the above bounds are tight.

Corollary 6. *The cardinality of every boundary defensive k-alliance S in the complete graph of order n is $|S| = \frac{n+k+1}{2}$.*

Notice that the complete graph $G = K_n$ has boundary defensive k-alliances if and only if $n + 1 + k$ is even.

It is well-known that the second-smallest Laplacian eigenvalue of a graph, frequently called the *algebraic connectivity*, is probably the most important information contained in the Laplacian spectrum. Also, the Laplacian spectral radius (the largest Laplacian eigenvalue of a graph) contains important information about the graph. These eigenvalues are related to several important graph invariants and they impose reasonably good bounds on the values of several parameters of graphs which are very hard to compute.

The algebraic connectivity of G, μ, and the Laplacian spectral radius, μ_*, satisfy the following equalities shown by Fiedler [4]:

$$\mu = 2n \min\left\{ \frac{\sum_{v_i\sim v_j}(w_i - w_j)^2}{\sum_{v_i\in V}\sum_{v_j\in V}(w_i - w_j)^2} \right\} \tag{4}$$

and

$$\mu_* = 2n \max\left\{ \frac{\sum_{v_i\sim v_j}(w_i - w_j)^2}{\sum_{v_i\in V}\sum_{v_j\in V}(w_i - w_j)^2} \right\}, \tag{5}$$

where not all the components of the vector $(w_1, w_2, \ldots, w_n) \in \mathbb{R}^n$ are equal.

The following theorems show the relationship between the algebraic connectivity (and the Laplacian spectral radius) of a graph and the cardinality of its boundary defensive k-alliances.

Theorem 7. *Let G be a connected graph G. If S is a boundary defensive k-alliance in G, then*

$$\left\lceil \frac{n\left(\mu - \left\lfloor \frac{d_1 - k}{2} \right\rfloor\right)}{\mu} \right\rceil \le |S| \le \left\lfloor \frac{n\left(\mu_* - \left\lceil \frac{d_n - k}{2} \right\rceil\right)}{\mu_*} \right\rfloor.$$

Proof. As S is a boundary defensive k-alliance in G,

$$\delta_{\bar{S}}(v) = \frac{d(v) - k}{2} \ge \left\lceil \frac{d_n - k}{2} \right\rceil, \quad \forall v \in S. \tag{6}$$

By Eq. (5), taking $w \in \mathbb{R}^n$ defined as

$$w_i = \begin{cases} 1 & \text{if } v_i \in S; \\ 0 & \text{otherwise}, \end{cases} \tag{7}$$

we have

$$\mu_* \geq \frac{n \sum_{v \in S} \delta_{\bar{S}}(v)}{|S|(n - |S|)}. \tag{8}$$

Then, by using (6) the above relation leads to

$$\mu_* \geq \frac{n \left\lceil \frac{d_n - k}{2} \right\rceil}{n - |S|}. \tag{9}$$

Therefore, solving (9) for $|S|$ and considering that it is an integer, we obtain the upper bound.

On the other hand,

$$\delta_{\bar{S}}(v) = \frac{d(v) - k}{2} \leq \left\lfloor \frac{d_1 - k}{2} \right\rfloor, \quad \forall v \in S. \tag{10}$$

Then, the lower bound is obtained as above by using (10) and (4) instead of (6) and (5), respectively. $\square$

If $G = K_n$, then $\mu = \mu_* = n$ and $d_1 = d_n = n - 1$. Therefore, the above theorem leads to the same result as Corollary 6.

The following result, given by Fiedler in [5], gives another relationship between the algebraic connectivity μ and the minimum and maximum degrees of the graph, which will be used to obtain bounds on the cardinality of boundary defensive k-alliances.

Lemma 8 ([5]). *Let G be a graph of order n; then $\mu \leq \frac{n}{n-1} d_n$ and $\mu_* \geq \frac{n}{n-1} d_1$.*

Theorem 9. *Let G be a connected graph. If S is a boundary defensive k-alliance in G, then*

$$\left\lceil \frac{n(\mu + k + 2) - \mu}{2n} \right\rceil \leq |S| \leq n - \left\lceil \frac{n(\mu - k) - \mu}{2n} \right\rceil.$$

Proof. As S is a boundary defensive k-alliance in G,

$$d_n \leq d(v) = 2\delta_S(v) - k \leq 2(|S| - 1) - k, \quad \forall v \in S, \tag{11}$$

and

$$d_n \leq d(v) = 2\delta_{\bar{S}}(v) + k \leq 2(n - |S|) + k, \quad \forall v \in S. \tag{12}$$

By Lemma 8, we have

$$\mu \leq \frac{n}{n-1} d_n. \tag{13}$$

Therefore, by using (11) and (12) in (13) we obtain both bounds. $\square$

Notice that in the case of the complete graph $G = K_n$, the above theorem leads to Corollary 6.

3. Boundary defensive k-alliances and planar subgraphs

The Euler formula states that for a connected planar graph of order n, size m and f faces, $n - m + f = 2$.

As a direct consequence of Theorem 4 and the Euler formula we obtain the following result.

Corollary 10. *Let $G = (V, E)$ be a graph and let $S \subset V$. Let c be the number of edges of G with one endpoint in S and the other endpoint outside of S. If S is a boundary defensive k-alliance in G such that $\langle S \rangle$ is planar connected with f faces, then*

(i) $|S| = \frac{c+4-2f}{2-k}$, *for* $k \neq 2$.

(ii) *If G is a d-regular graph, then* $|S| = \frac{4f-8}{d+k-4}$ *and* $c = \frac{2(d-k)(f-2)}{d+k-4}$, *for* $k \in \{5 - d, \ldots, d\}$.

Theorem 11. *Let G be a graph and let S be a boundary defensive k-alliance in G such that $\langle S \rangle$ is planar connected with f faces; then*

$$|S| \leq \left\lfloor \frac{\sqrt{16 - 8f + (n + k - 2)^2} + n + k - 2}{2} \right\rfloor.$$

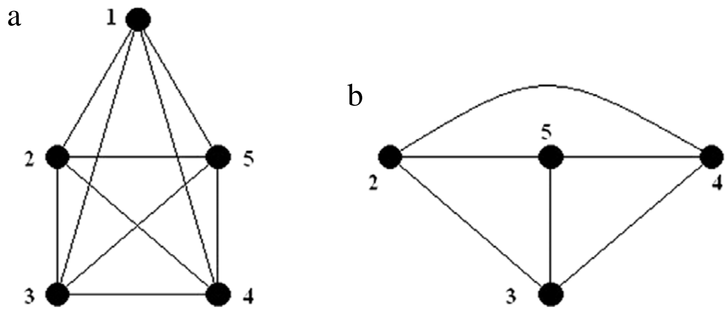

Fig. 2. (a) The complete graph $G = (V, E) \simeq K_5$ is an example of a 4-regular graph where the set $S = \{2, 3, 4, 5\} \subset V$ is a boundary defensive (2)-alliance. (b) $\langle S \rangle \simeq K_4$ is planar with four faces. In this case $|S| = \frac{4f-8}{d+k-4}$.

Proof. If S denotes a boundary defensive k-alliance in G, then

$$\sum_{v \in S} \delta_S(v) = \sum_{v \in S} \delta_{\bar{S}}(v) + k|S| \leq |S|(n - |S|) + k|S|.$$

By the Euler formula on $\langle S \rangle$ we have $\sum_{v \in S} \delta_S(v) = 2(|S| + f - 2)$, so the result follows. □

The above bound is tight. For instance, the bound is attained for the complete graph $G = K_5$ where the set $S = \{2, 3, 4, 5\}$ forms a boundary defensive 2-alliance and $\langle S \rangle \simeq K_4$ is planar with $f = 4$ faces (Fig. 2).

Theorem 12. *Let G be a graph and let S be a boundary defensive k-alliance in G such that $\langle S \rangle$ is planar connected with $f > 2$ faces.*

- *If $k \in \{5 - d_1, \ldots, d_1\}$, then* $|S| \geq \left\lceil \frac{4f-8}{d_1+k-4} \right\rceil$,
- *If $k \in \{5 - d_n, \ldots, d_1\}$, then* $|S| \leq \left\lfloor \frac{4f-8}{d_n+k-4} \right\rfloor$.

Proof. As S is a boundary defensive k-alliance in G,

$$\sum_{v \in S} \delta_S(v) = \sum_{v \in S} \delta_{\bar{S}}(v) + k|S|.$$

Hence,

$$|S|\frac{d_n - k}{2} + k|S| \leq \sum_{v \in S} \delta_S(v) \leq |S|\frac{d_1 - k}{2} + k|S|. \tag{14}$$

Therefore, by the Euler formula on $\langle S \rangle$ and the above inequalities, the bounds on $|S|$ follow. □

By Corollary 10, the above bounds are tight.

4. Partitioning a graph into r boundary defensive k-alliances

Let $G = (V, E)$ be a graph and let $\wp = \{X_1, X_2, \ldots, X_r\}$ be a partition of V into r boundary defensive k-alliances. Suppose $x = \max_{1 \leq i \leq r}\{|X_i|\}$ and $y = \min_{1 \leq i \leq r}\{|X_i|\}$, $\frac{n}{x} \leq r \leq \frac{n}{y}$. Examples of bounds of r are the following two corollaries.

As a consequence of Theorem 5 we obtain the following bounds.

Corollary 13. *If G can be partitioned into r boundary defensive k-alliances, then* $\frac{2n}{2n-d_n+k} \leq r \leq \frac{2n}{d_n+k+2}$.

The above bounds are tight. For instance, from the above result we obtain that the complete graph $G = K_n$ can be partitioned into $r = \frac{2n}{n+k+1}$ boundary defensive k-alliances. In particular, if n is even, each pair of vertices of K_n forms a boundary defensive $(3 - n)$-alliance. Thus, K_n can be partitioned into $\frac{n}{2}$ of these alliances. Moreover, the upper bound is attained, for instance, in the case of $G = K_{t_1} \times C_{t_2}$, where C_{t_2} denotes a cycle of order t_2. In such a case, G is a $(t_1 + 1)$-regular graph of order $n = t_1 t_2$. Thus, for $k = t_1 - 3$ we obtain $r = t_2$. Notice that each one of the t_2 copies of K_{t_1} is a boundary defensive $(t_1 - 3)$-alliance in G.

Remark 14. The complete graph of order n, $G = K_n$, can be partitioned into r boundary defensive k-alliances if, and only if, $n \equiv 0(r)$ and $k = \frac{2n}{r} - n - 1$.

As a consequence of Theorem 7 we obtain the following result.

Corollary 15. *If G can be partitioned into r boundary defensive k-alliances, then* $\frac{2\mu_*}{2\mu_*-d_n+k} \leq r \leq \frac{2\mu}{2\mu-d_1+k}$.

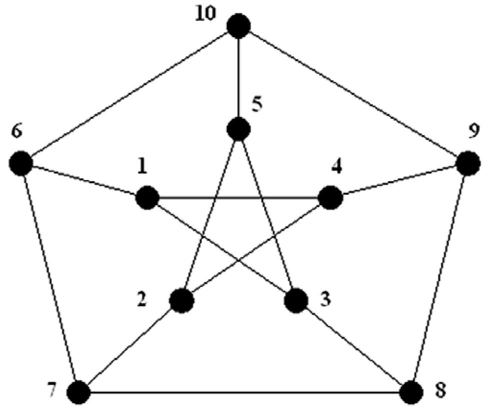

Fig. 3. The sets $S_1 = \{1, 2, 3, 4, 5\}$ and $S_2 = \{6, 7, 8, 9, 10\}$ form a partition of the Petersen graph into two boundary defensive (1)-alliances.

The above bounds are tight. An example where the bounds are attained is the case of the complete graph $G = K_n$. Moreover, by Corollary 15 we conclude, for instance, that if the Petersen graph (Fig. 3) can be partitioned into r boundary defensive k-alliances, then $k = 1$ and $r = 2$ (in this case $d_1 = d_2 = 3$, $\mu = 2$ and $\mu_* = 5$).

Theorem 16. *Let $G = (V, E)$ be a graph and let $M \subset E$ be a cut set partitioning V into two boundary defensive k-alliances S and $\bar{S}$, where $k \neq d_1$ and $k \neq d_n$. Then* $\left\lceil \frac{2m-kn}{2(d_1-k)} \right\rceil \leq |S| \leq \left\lfloor \frac{2m-kn}{2(d_n-k)} \right\rfloor$ *and* $|M| = \frac{2m-kn}{4}$.

Proof. As S is a boundary defensive k-alliance in G,

$$\sum_{v\in S} d(v) = 2\sum_{v\in S} \delta_{\bar{S}}(v) + k|S|.$$

Moreover, as $\bar{S}$ is a boundary defensive k-alliance in G,

$$\sum_{v\in \bar{S}} d(v) = 2\sum_{v\in \bar{S}} \delta_{S}(v) + k(n-|S|).$$

Therefore, from these two equalities we obtain

$$2m = 4\sum_{v\in S} \delta_{\bar{S}}(v) + kn. \tag{15}$$

So, we have $|M| = \sum_{v\in S} \delta_{\bar{S}}(v) = \frac{2m-nk}{4}$. Moreover, by using (6) and (10) in (15), we obtain the bounds on $|S|$. □

Corollary 17. *Let $G = (V, E)$ be a d-regular graph and let $M \subset E$ be a cut set partitioning V into two boundary defensive k-alliances S and $\bar{S}$. Then $|S| = \frac{n}{2}$ and $|M| = \frac{n(d-k)}{4}$.*

Theorem 18. *If $\{X, Y\}$ is a partition of V into two boundary defensive k-alliances in $G = (V, E)$, then, without loss of generality,*

$$\left\lceil \sqrt{\frac{n(kn-2m+n\mu)}{4\mu}} + \frac{n}{2} \right\rceil \leq |X| \leq \left\lfloor \sqrt{\frac{n(kn-2m+n\mu_*)}{4\mu_*}} + \frac{n}{2} \right\rfloor$$

and

$$\left\lceil \frac{n}{2} - \sqrt{\frac{n(kn-2m+n\mu_*)}{4\mu_*}} \right\rceil \leq |Y| \leq \left\lfloor \frac{n}{2} - \sqrt{\frac{n(kn-2m+n\mu)}{4\mu}} \right\rfloor.$$

Proof. By Theorem 16,

$$\sum_{v\in X} \delta_Y(v) = \frac{2m-kn}{4}. \tag{16}$$

Moreover, as we have shown in the proof of Theorem 7,

$$\mu \leq \frac{n\sum_{v\in X} \delta_Y(v)}{|X|(n-|X|)} \leq \mu_*. \tag{17}$$

Therefore, by using (16) in both sides of (17) we obtain the bounds on $|X|$ and $|Y| = n - |X|$. $\square$

The above bounds are tight. For instance, in the case of the complete graph $G = K_n$, the above theorem leads to $|X| = \frac{n}{2} + \sqrt{\frac{n(k+1)}{4}}$ and $|Y| = \frac{n}{2} - \sqrt{\frac{n(k+1)}{4}}$. By using Remark 14, we have $k = -1$ and, as a consequence, $|X| = |Y| = \frac{n}{2}$.

By Corollary 17 and Theorem 18 we obtain the following interesting consequence.

Theorem 19. *Let $G = (V, E)$ be a d-regular graph. If V has a partition into two boundary defensive k-alliances, then the algebraic connectivity of G is $\mu = d - k$ (an even number).*

By the above necessary condition of existence of a partition of V into two boundary defensive k-alliances we obtain, for instance, that the icosahedron cannot be partitioned into two boundary defensive k-alliances, because its algebraic connectivity is $\mu = 5 - \sqrt{5} \notin \mathbb{Z}$. Moreover, the Petersen graph can only be partitioned into two boundary defensive k-alliances for the case of $k = 1$, because $d = 3$ and $\mu = 2$.

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